LECTURE XXII

1. Introduction

1.1. **Motivating Example.** Consider the sum of numbers

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$$

We are adding powers of $\frac{1}{2}$, until the end of life. Does this sum converge, i.e. add up to a finite number? If so, is there a nice formula to find the sum of other sum of powers? Or does it diverge to infinity?

Poll: Finite (True) or Infinite (False)?

The above is an example of infinite series. The word "**infinite**" means there are infinite number of terms. The word "**series**" means that one is adding up all those terms.

We say an infinite series **converges** when it yields a finite real number. We say an infinite series **diverges** when it is **NOT convergent**. It does not necessarily mean it is infinity. This notion will be more clarified once we connect sequence and series.

To combine with what we learn from before about sequences, we note that the above series is a sum of a **sequence** no other than

$$a_n = \frac{1}{2^n}, n = 0, 1, 2, \dots$$

Then, to have a short hand notation for the above series, we write

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = \sum_{n=0}^{\infty} \frac{1}{2^n}$$

note that the series starts at $n = 0$. **You must always check whether the short hand sigma notation and its bounds match with the series you are representing.** For example, a series like

$$\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots = \sum_{n=1}^{\infty} \frac{1}{3^n}$$

now starts with $n = 1$. You will do more exercises during recitation.

In general, we write an infinite series as

$$\sum_{n=1}^{\infty} a_n$$

where a_n is a specified sequence.

1.2. **Technique of Reindexing.** Poll: How do we rewrite $\sum_{n=0}^{\infty} \frac{1}{2^n}$ so that n starts at 1 instead of 0, without changing the terms of the series?

$$A : \sum_{n=1}^{\infty} \frac{1}{2^{n+1}}$$

$$B : \sum_{n=1}^{\infty} \frac{1}{2^{n-1}}$$

One sanity check after you made your choice is to check whether the first term is the same as that of the original. For $a_n = \frac{1}{2^n}, n = 0, 1, 2, \dots$, choice A gives you its 1st term $\frac{1}{2^{1+1}} = \frac{1}{4} \neq 1 = a_0$ yet B gives you $\frac{1}{2^{1-1}} = 1$ which is more believable.

Date: March. 23rd, 2020.

A common technique sometimes used to compare series (which we learn in a later section) is to shift the index. Take the same example above, we can start the series at $n = 2$ and shift the index of the summand from a_n to a_{n-1} namely,

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = \sum_{n=2}^{\infty} \frac{1}{2^{n-2}}$$

(note the original sequence is $a_n = \frac{1}{2^n}$ for n starting at 0, now to shift by 2 units on n, a_n will become $a_{n-2} = \frac{1}{2^{n-2}}$, as stated.)

1.3. **Sequence of Partial Sums.** To analyze the convergent/divergent behaviour of a series, we often consider the partial sum of a series, that is,

$$S_N = \sum_{n=1}^{N} a_n$$

namely, we are only adding up to N terms of the sequence. **Now, S_N itself is a sequence in the index** N. By definition of S_N, we see that

$$\lim_{N\to\infty} S_N = \lim_{N\to\infty} \sum_{n=0}^{N} a_n = \sum_{n=0}^{\infty} a_n$$

i.e., **the limit of the sequence of partial sums S_N is the value of the series**. If the sequence S_N converges to some finite value L, then the series is equal to L, i.e.

$$\sum_{n=0}^{\infty} a_n = \lim_{N\to\infty} S_N = L$$

We thus have reduced the study of a series down to the study of the convergence of its sequence of partial sums. If we can find a nice formula for S_N, we can study the convergence/divergence properties of the series with even more ease.

Example. Take the $a_n = \frac{1}{2^{n-1}}$ as an example.

$$\sum_{n=1}^{\infty} \frac{1}{2^{n-1}}$$

The partial sum is

$$S_N = \sum_{n=1}^{N} \frac{1}{2^{n-1}}$$

In fact, for each N, at least for N small, we can write out the first few terms of S_N

$$S_1 = 1$$

$$S_2 = \sum_{n=0}^{1} \frac{1}{2^{n-1}} = 1 + \frac{1}{2} = \frac{3}{2}$$

$$S_3 = \sum_{n=0}^{2} \frac{1}{2^{n-1}} = 1 + \frac{1}{2} + \frac{1}{4} = \frac{7}{4}$$

$$S_4 = \sum_{n=0}^{3} \frac{1}{2^{n-1}} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{15}{8}$$

$$\vdots$$

Asking whether the sequence S_N converges is equivalent to asking whether $\sum_{n=1}^{\infty} \frac{1}{2^{n-1}}$ yields a finite number.

1.4. **A Special Riemann Sum Interpretation of a Series.** For a series

$$\sum_{n=1}^{\infty} a_n$$

define a function

$$f(x) = a_n, \quad n-1 \le x \le n, \quad n = 1, 2, \ldots$$

This series is no different from a Riemann sum of f with subinterval length $\Delta x = 1$. Let's draw a picture for, say the series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

(called the Harmonic series). Here, the function that corresponds to this series is the step function

$$f(x) = \frac{1}{n}, \quad n-1 \le x < n, \quad n = 1, 2, \ldots$$

(try plotting this, it should be boxes of equal width but descending heights. If you don't know how to start, plug in $n = 1$, graph $f(x) = 1, 0 \le x \le 1$, and go on for each n).

This technique comes in handy when we try to check if a series is convergent. This particular series is related to the divergent integral

$$\int_1^{\infty} \frac{1}{x} dx$$

but not exactly the same. We will see this more precisely in a later section (integral comparison test).

2. Special Series

2.1. **Geometric series.** For a number $a \neq 0$ and a ratio r, consider the sum

$$a + ar + ar^2 + \cdots = a \sum_{n=1}^{\infty} r^{n-1}$$

For what r does this series converge? Suppose $r = 1$.

$$\sum_{n=1}^{\infty} r^{n-1} = 1 + 1 + 1 + \cdots = \infty$$

which makes sense. Furthermore, for $r \ge 1$, by "comparison",

$$\sum_{n=1}^{\infty} r^{n-1} \ge \sum_{n=1}^{\infty} 1 = \infty$$

In fact, the series converges for $|r| < 1$, or $-1 < r < 1$ (note the possibility of negative ratios). There is a beautiful formula for the series with $|r| < 1$,

$$\sum_{n=1}^{\infty} r^{n-1} = \frac{1}{1-r}$$

In other words, we end up with results like

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1} = \frac{1}{1-\frac{1}{2}} = 2$$

$$1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots = \sum_{n=1}^{\infty} \left(-\frac{1}{2}\right)^{n-1} = \frac{1}{1-\left(-\frac{1}{2}\right)} = \frac{2}{3}$$

For $|r| \ge 1$, the formula no longer makes sense. Say, for $r = 2$, the formula would read

$$1 + 2 + 2^2 + 2^3 + \cdots = \frac{1}{1-2} = -\frac{1}{2}$$

which is garbage.